DAMPING OF PLASMA WAVES

R. K. Mazitov

Zhurnal prikladnoi mekhaniki i tekhnicheskoi fiziki, No. 1, pp. 27-31, 1965

The mechanism of formation of a plateau of the electron velocity distribution function in monochromatic plasma wave damping is discussed. It is shown that the distribution function is subject to strong modulation in a neighborhood of the phase velocity of the wave and that the steady state is established as a result of collisions.

The collisionless damping obtained in the linear approximation in [1] is caused by resonance particles and depends on the electron velocity distribution function in the region

$$v_p - \sqrt{2e\Phi_0/m} \leqslant v_p \leqslant v + \sqrt{2e\Phi_0/m} \qquad (0.1)$$

where Φ_0 is the amplitude of the wave field potential, and v_p is the phase velocity.

According to [2, 3], where the so-called quasilinear approximation is used, when $t \to \infty$ the distribution function in a neighborhood of the phase velocity assumes the form of a "plateau" and damping ceases.

Since the quasilinear approximation is valid only for a sufficiently wide wave packet, it is interesting to discover how the steady-state is achieved in the case of a monochromatic wave.

In this case when $t \to \infty$ the distribution function suffers powerful modulation, as indicated by L. D. Landau [1], who gives the asymptotic form of this function:

$$f = f_0 + f_1 e^{-ikvt}. \qquad (0.2)$$

It is shown below that in a nonlinear treatment the nature of the modulation of the distribution function in the resonance region depends essentially on the electric field. The process of "plateau" formation is investigated on the basis of qualitative considerations. In the case of a monochromatic wave this plateau is established only by taking into account collisions. The time-dependence of the wave energy is also derived.

1. We shall assume the wave profile to be given and neglect its distortion. It will be insignificant if the number of electrons trapped by the wave field and the wave amplitude are small. Assuming the distribution of the trapped electrons at the instant t = 0 to be Maxwellian, we shall write the condition of weak distortion of the wave profile in the form:

$$\sqrt{2e\Phi_0/m} \ll v_T \ll v_p \qquad (1.1)$$

where v_T is the average thermal velocity of the electrons.

In order to simplify the calculations, we shall limit the expansion of the distribution function at the instant t = 0 and the point $v = \omega/k$ to the first two terms

$$f_{t=0} = f_0(w/k) + (v - w/k) f_0'(w/k). \qquad (1.2)$$

Thus, in order to find the distribution function it is necessary to solve the kinetic equation, whose coefficients in the system of coordinates associated with the wave are independent of time:

$$\frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} + \frac{e\Phi'(x)}{m}\frac{\partial f}{\partial u} = \operatorname{st} f, \qquad (1.3)$$

with the initial condition

$$f_{t=0} = f_0(w/k) + u f_0'(w/k). \qquad (1.4)$$

2. We shall first consider the potential profile shown in Fig. 1. The trapped electrons with a velocity greater than the phase velocity overtake the wall cd and are reflected back, giving up energy to the wave; electrons with a velocity less than the phase velocity are accelerated by the overtaking wall ab, receiving energy from the wave. The change in the kinetic energy of the electrons is equal to the change in the energy of the wave, taken with the opposite sign:

$$\Delta T = -{}^1/_2 \lambda \Delta\varepsilon, \qquad (2.1)$$

where T is the kinetic energy of the trapped electrons, and ε is the energy of the wave referred to unit length.

Let us write out the expression for T:

$$T=\int\limits_{-\frac{1}{2}\lambda}^{+\frac{1}{2}\lambda}\int\limits_{u_-}^{u_+}\left\{f(x,u,0)\left[\frac{mu^2}{2}+\frac{m}{2}\left(\frac{\omega}{k}\right)^2\right]+f(x,u,t)\,mu\frac{\omega}{k}\right\}du\,dx\ \left(u_\pm=\pm\sqrt{2e\Phi_0/m}\right). \tag{2.2}$$

We shall denote the first term by T_∞. Below we show that this is the kinetic energy of the electrons when $t\to\infty$. The second term will be a linear function of time:

$$\Delta T=T-T_\infty=(2e\Phi_0/m)^2\left({}^2/_3\,t_1-t\right)f_0' \quad \text{for } t\leqslant t_1=\lambda/2u_+. \tag{2.3}$$

Hence,

$$\Delta\varepsilon=\varepsilon-\varepsilon_\infty=-2\lambda^{-1}(2e\Phi_0/m)^2\left({}^2/_3t_1-t\right)f_0'. \tag{2.4}$$

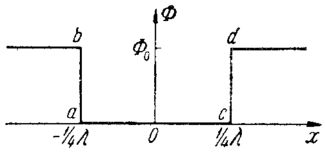

Fig. 1.

Note that the damping decrement $(1/2)\varepsilon\ d\varepsilon/dt$, obtained in [2] by means of a simple consideration of the energy transfer between the wave and the trapped electrons, agrees correct to the numerical factor with the decrement calculated by L. D. Landau [1].

Omitting the rather cumbersome calculations, we shall give the expression for the wave energy:

$$\varepsilon=\varepsilon_\infty+2\,(2e\Phi_0/m)^{3/2}f_0'\Psi(n)\quad \text{when } t\geqslant t_1 \tag{2.5}$$

where

$$\Psi(n)=\begin{cases}-\left[n^2(\gamma^2-\gamma)+{}^7/_6\,n\gamma^3+{}^1/_2\,\gamma^4-{}^1/_3\,\gamma^3\right](n+\gamma)^{-3} & (n=1,3,\ldots)\\ \left[n^2(\gamma^2+\gamma)+{}^2/_3\,n\gamma^3+{}^1/_2\gamma^4-{}^1/_3\gamma^3\right](n+\gamma)^{-3} & (n=2,4,\ldots)\end{cases}$$

$$t=t_1(n+\gamma),\quad n=[t/t_1].$$

If $n\to\infty$, then $\Psi(n)\to 0$. Consequently, ε_∞ is the energy of the wave when $t\to\infty$. The relation between ε and time is shown in Fig. 2.

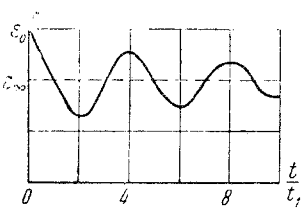

Fig. 2.

As $t\to\infty$ the distribution function of the trapped electrons suffers powerful modulation. The function $f(x,u,t)$ for fixed u has only two values: $f_-=f_0-uf_0'$ and $f_+=f_0+uf_0'$. Consequently, for $x>0$ it can be put in the form:

$$f=\begin{cases}f_+ & \left(0<u<(x+{}^1/_4\,\lambda)\,t^{-1},\ [x+{}^1/_4\,\lambda+(2m+1)\,{}^1/_2\,\lambda]\,t^{-1}<u<[x+{}^1/_4\lambda+\right.\\ & \left.\quad+(2m+2)\,{}^1/_2\lambda]\,t^{-1}\right)\\ f_- & \left([x+{}^1/_4\lambda+2m{}^1/_2\lambda]\,t^{-1}<u<[x+{}^1/_4\lambda+(2m+1)\,{}^1/_2\lambda]\,t^{-}\right)\\ f_+ & \left(({}^1/_4\lambda-x)\,t^{-1}<u<0,\ -[{}^1/_4\lambda-x+(2m+2)\,{}^1/_2\,\lambda]\,t^{-1}<u<-[{}^1/_4\lambda-\right.\\ & \left.\quad-x+(2m+1)\,{}^1/_2\,\lambda]\,t^{-1}\right)\\ f_- & \left(-[{}^1/_4\lambda-x+(2m+1)\,{}^1/_2\,\lambda]\,t^{-1}<u<-[{}^1/_4\lambda-x+2m{}^1/_2\lambda]\,t^{-1}\right)\end{cases}$$
$$(m=0,1,2,\ldots). \tag{2.6}$$

The last intervals in Eq. (2.6) have been omitted in order not to make the formulas too cumbersome. f may be written in the same way for $x<0$, too.

The modulation of the distribution function is caused by the difference in the periods of the electrons. We shall extend the results obtained to potential profiles with somewhat smoothed angles and assume that $\Phi'(\pm\lambda/4)\neq\infty$. Let us investigate the effect of collisions on the form of the distribution function. We shall take the term describing the collisions in the form [4]:

$$\mathrm{st}\,f=\nu\frac{\partial}{\partial u}\left[v_T{}^2\frac{\partial f}{\partial u}+(v_\mathrm{p}+u)\,f\right]. \tag{2.7}$$

Here ν has the meaning of collision frequency. When $t\to\infty$ the first term in (2.7) increases strongly, so that the second term can be neglected. The collisions become important for all x when

$$\frac{e\Phi'_\mathrm{max}}{m}\frac{\partial f}{\partial u}\sim\nu v_T{}^2\frac{\partial^2 f}{\partial u^2}\quad\text{or}\quad t\sim t_1\frac{e\Phi'_\mathrm{max}}{m}\frac{\sqrt{2e\Phi_0/m}}{\nu v_T{}^2}. \tag{2.8}$$

The result of diffusion in the velocity space is that the oscillating distribution function "broadens out" and at the point $u=0$ ($v=\omega/k$) when $t\to\infty$ it assumes the form of a plateau.

3. We shall now consider the case of a sinusoidal wave

$$\Phi = {}^1/_2\Phi_0(1-\cos kx)\,. \tag{3.1}$$

In order to solve Eq. (1.3) it is necessary to determine the integrals of motion of an electron in the wave field. By means of the substitution

$$\sin\xi = \frac{\sin {}^1/_2\, kx}{\sin {}^1/_2\, kx_0} \qquad \left(\sin {}^1/_2\, kx_0 = \frac{E}{e\Phi_0}\right) \tag{3.2}$$

the integral

$$\int_{x_1}^{x} \frac{dx}{\sqrt{2m^{-1}\left[E - e\Phi_0\,(\sin {}^1/_2\, kx)^2\right]}} = t \qquad (x_1 - \text{initial coordinate}) \tag{3.3}$$

is made to assume the simpler form:

$$\int_{\xi}^{\xi_1} \frac{d\xi}{\sqrt{1-\alpha^2\sin^2\xi}} = t\,\frac{k\sqrt{2e\Phi_0/m}}{2} \qquad (\alpha = \sin {}^1/_2\, kx_0)\,. \tag{3.4}$$

At the instant $t = 0$

$$f(x_1, E) = \frac{f_0 \pm \sqrt{2m^{-1}\left[E - e\Phi_0\,(\sin {}^1/_2\, kx_1)^2\right]}}{m\sqrt{2m^{-1}\left[E - e\Phi_0\,(\sin {}^1/_2\, kx_1)^2\right]}}\,. \tag{3.5}$$

Here, the plus is for $u > 0$ and minus for $u < 0$. At any other instant t the distribution function will differ from (3.5) in that in the numerator $x_1 = x_1(x, E, t)$, while in the denominator instead of x_1 we get x.

Using the integral (3.4), we transform the expression for the distribution function:

$$f(x, E, t) = \begin{cases} \dfrac{f_0 \pm f_0'\alpha\sqrt{2e\Phi_0/m}\,\mathrm{cn}\,[\tau \pm \chi(\xi)]}{m\sqrt{2m^{-1}\left[E - e\Phi_0\,(\sin {}^1/_2\, kx)^2\right]}} & \left(-\dfrac{\pi}{2} < kx < 0\right) \\[2ex] \dfrac{f_0 \pm f_0'\alpha\sqrt{2e\Phi_0/m}\,\mathrm{cn}\,[\tau \mp \chi(\xi)]}{m\sqrt{2m^{-1}\left[E - e\Phi_0\,(\sin {}^1/_2\, kx)^2\right]}} & \left(0 < kx < \dfrac{\pi}{2}\right) \end{cases} \tag{3.6}$$

with the following notation:

$$\tau = t\,\frac{k\sqrt{2e\Phi_0/m}}{2}\,, \qquad \chi(\xi) = \int_0^{\xi} \frac{d\xi}{\sqrt{1-\alpha^2\sin^2\xi}}$$

where α is the modulus of the elliptic function. Note that if we go over to the variables x, u, t, then, for fixed x, f will be determined as a function of u on the interval:

$$(-u_0\cos {}^1/_2\, kx,\; u_0\cos {}^1/_2\, kx) \qquad (u_0 = \sqrt{2e\Phi_0/m})\,. \tag{3.7}$$

As in the first case, the distribution function suffers powerful modulation at large values of τ, because the periods of the electrons in the wave field depend on the energy E. The shape of the distribution function for the bottom of the potential well is given in Fig. 3. The maximum frequency of the oscillations f for fixed x will occur at the ends of the interval $(u_0\cos(1/2)kx,\ u_0\cos(1/2)kx)$, where the collisions therefore become important earlier than for $u \approx 0$. Thus, the distribution function is first smoothed near the ends of the interval, but in the course of time in the middle too, i.e., at the point $u = 0$ $(v = \omega/k)$.

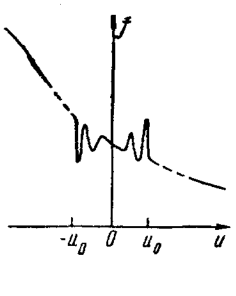

Fig. 3.

We shall now find the kinetic energy of the electrons trapped by the wave field:

$$T = \int_{-{}^1/_2\lambda}^{{}^1/_2\lambda}\int_0^{e\Phi_0}\left\{f(x, E, 0)\left[\frac{mu^2}{2} + \frac{m}{2}\left(\frac{\omega}{k}\right)^2\right] + f(x, E, t)\,u\,\frac{\omega}{k}\right\}\frac{dE\,dx}{mu}\,. \tag{3.8}$$

Here

$$u = \sqrt{2m^{-1}\left[E - e\Phi_0\,(\sin {}^1/_2\, kx)^2\right]}\,.$$

We substitute (3.6) in (3.8) and write out the second term, which we denote by ΔT:

$$\Delta T = -8\,\frac{\omega}{k}\left(\frac{2e\Phi_0}{m}\right)^{1/2}\frac{e\Phi_0}{k}\,f_0'\int_0^1 \alpha^3\,\mathrm{cn}\,\tau\int_0^{\pi/2}\frac{\cos^2\xi\,d\xi}{[1-\alpha^2(\sin\xi)^2(\mathrm{sn}\,\tau)^2]\sqrt{1-\alpha^2(\sin\xi)^2}}\;. \qquad (3.9)$$

It is clear that the first term in (3.8) is the kinetic energy of the electrons when $\tau \to \infty$.

After expanding

$$[1-\alpha^2(\sin\xi)^2(\mathrm{sn}\,\tau)^2]^{-1}$$

and integrating with respect to ξ, we obtain

$$\Delta T = -8\left(\frac{2e\Phi_0}{m}\right)^{1/2}\frac{e\Phi_0}{k}f_0'\int_0^1\alpha^3\,\mathrm{cn}\,\tau\left[\sum_{j=0}^{\infty}B_j(\alpha)\,\mathrm{sn}^{3j}\tau\right]d\alpha\;. \qquad (3.10)$$

Here the $B_j(\alpha)$ ($j = 0, 1, 2\ldots$) are expressed in terms of complete elliptic integrals.

A rough estimate shows that the contribution of the first term in (3.10) is approximately four times greater than the contribution of the rest. Consequently, we shall consider the integral:

$$J = \int_0^1 \alpha^3 B_0\,\mathrm{cn}\,\tau\,d\alpha = \int_0^1 K'(\alpha)\,\alpha^2(1-\alpha^2)\,\mathrm{cn}\,\tau\,d\alpha\;. \qquad (3.11)$$

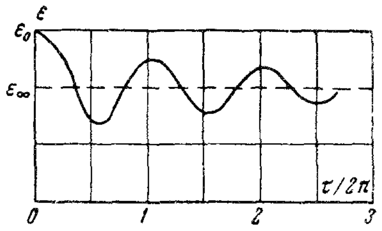

Fig. 4.

Here $K(\alpha)$ is a complete elliptic integral of the first kind. We make the change of variables:

$$J = K(0)\int_0^1\frac{\alpha^2(y)\,[1-\alpha^2(y)]}{y^2}\,\mathrm{cn}\,\tau\,dy, \qquad \frac{K(0)}{K(\alpha)} = y \qquad (3.12)$$

Let us investigate the asymptotic behavior of J as $\tau \to \infty$. Noting that

$$\frac{\alpha^2(1-\alpha^2)}{y^2} \sim \frac{1}{y^2}e^{-1/y} \text{ when } y \to 0, \qquad \mathrm{cn}\,\tau \sim \cos\tau y \text{ when } y \to 1$$

and integrating (3.12) by parts, we obtain:

$$J = 4K(0)\,[\tau^{-2}\cos\tau + O(\tau^{-3})]\;. \qquad (3.13)$$

The second term will be small in comparison with the first when $\tau > 10$. Using the relation

$$\Delta T = -\lambda\Delta\varepsilon \qquad (3.14)$$

we find an expression for the wave energy at large values of τ:

$$\varepsilon = \varepsilon_\infty + 8\,\frac{\omega}{k}\,\frac{e\Phi_0}{k}\left(\frac{2e\Phi_0}{m}\right)^{1/2}\frac{\cos\tau}{\tau^2}\;. \qquad (3.15)$$

The time dependence of the wave energy is shown in Fig. 4.

In conclusion the authors wish to express their thanks to R. Z. Sagdeev for guidance and to V I. Karpman for discussion of the results.

REFERENCES

1. L. D. Landau, "Concerning the oscillations of an electron plasma," Zh. eksperim. i teor. fiz., vol. 16, p. 574, 1946.

2. A. A. Vedenov, E. P. Velikhov, and R. Z. Sagdeev, "The stability of a plasma," Usp. fiz., vol. 73, p. 701, 1961.

3. A. A. Vedenov, E. P. Velikhov, and R. Z. Sagdeev, "Nonlinear oscillations of a rarefied plasma," Yadernyi sintez, vol. 1, p. 82, 1961.

4. V. E. Zakharov and V. I. Karpman, "Nonlinear theory of damping of plasma waves," Zh. eksperim. i teor fiz., vol. 43, p. 490, 1962.

10 March 1964 Novosibirsk